A1: $y^{(5)} - 3y^{(4)} + 5y''' - 5y'' + 4y' - 2y = g(x) \quad mit \quad g(x) \in \{6e^x, 3e^x \cdot \sin x\}$

a) Homogene Lösung:

$$P(\alpha) = \alpha^5 - 3\alpha^4 + 5\alpha^3 - 5\alpha^2 + 4\alpha - 2 = (\alpha - 1)\cdot(\alpha^2 + 1)\cdot(\alpha^2 - 2\alpha + 2) = 0$$

$$\Rightarrow \alpha_1 = 1; \quad \alpha_{2/3} = \pm i; \quad \alpha_{4/5} = 1 \pm i$$

$$y_H = C_1 \cdot e^x + C_2 \cdot \cos x + C_3 \cdot \sin x + e^x \cdot (C_4 \cdot \cos x + C_5 \cdot \sin x).$$

b) Inhomogene Lösung:

$g_1(x) = 6e^x \quad ; \quad c = 1 \quad ; \quad$ einfache Resonanz zu $\alpha_1 = 1$

$$\Rightarrow y_{P_1} = \frac{6e^x \cdot x}{P'(1)} = 3x \cdot e^x \quad (P'(\alpha) = 5\alpha^4 - 12\alpha^3 + 15\alpha^2 - 10\alpha + 4)$$

$g_2(x) = 3e^x \cdot \sin x \quad ; \quad c = 1 + i \quad ; \quad$ einfache Resonanz zu $\alpha_4 = 1 + i$

$$\Rightarrow y_{P_2} = \mathrm{Im}\left[\frac{3e^{(1+i)x} \cdot x}{P'(1+i)}\right] = 3x \cdot e^x \cdot \mathrm{Im}\left[\frac{\cos x + i \cdot \sin x}{-2(1+2i)}\right] =$$

$$-\frac{3x \cdot e^x \cdot \mathrm{Im}\left[\frac{(\cos x + i \sin x)\cdot(1-2i)}{(1+2i)\cdot(1-2i)}\right]}{2} = +\frac{3x \cdot e^x}{10}\cdot(2\cos x - \sin x)$$

c)

$$y_{\text{allg}} = C_1 \cdot e^x + C_2 \cdot \cos x + C_3 \cdot \sin x + e^x \cdot (C_4 \cdot \cos x + C_5 \cdot \sin x) + \frac{3x \cdot e^x}{10}\cdot(2\cos x - \sin x + 10)$$

Randwertproblem:

$$\left.\begin{array}{lll}(1) & y_H(0) = 0 & \Rightarrow \quad 0 = C_1 + C_2 + C_4 \\ (2) & y_H = const & \Rightarrow \quad 0 = C_1 = C_4 = C_5\end{array}\right\} \Rightarrow \quad C_2 = 0$$

$$(3) \quad y_H{}'(\pi) = 1 \quad \Rightarrow \quad 1 = -C_3$$

Also :

$$y_{\text{spez}} = -\sin x + \frac{3x \cdot e^x}{10}\cdot(2\cos x - \sin x + 10)$$

A2:

$y' + x \cdot y' - p \cdot y = 0 \quad ; \quad AWP : y(0) = 1$

$$Ansatz: \quad y = \sum_{n=0}^{\infty} a_n \cdot x^n \quad ; y' = \sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1}$$

$$\Rightarrow \quad (1+x)\cdot\sum_{n=1}^{\infty} n \cdot a_n \cdot x^{n-1} - \pi \cdot \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} a_n \cdot (n \cdot x^{n-1} + (n - \pi)\cdot x^n) - \pi \cdot a_0 = 0$$

Koeffizientenvergleich:

$$\begin{array}{lll}
x^0: & a_1\cdot 1-\pi\cdot a_0=0 & \Leftrightarrow a_1=\frac{\pi}{1}\cdot a_0=\binom{\pi}{1}\cdot a_0 \\
x^1: & a_2\cdot 2+(1-\pi)\cdot a_1=0 & \Leftrightarrow a_2=\frac{\pi\cdot(\pi-1)}{1\cdot 2}\cdot a_0=\binom{\pi}{2}\cdot a_0 \\
x^2: & a_3\cdot 3+(2-\pi)\cdot a_2=0 & \Leftrightarrow a_3=\frac{\pi\cdot(\pi-1)\cdot(\pi-2)}{1\cdot 2\cdot 3}\cdot a_0=\binom{\pi}{3}\cdot a_0 \\
x^3: & a_4\cdot 4+(3-\pi)\cdot a_3=0 & \Leftrightarrow a_4=\frac{\pi\cdot(\pi-1)\cdot(\pi-2)\cdot(\pi-3)}{1\cdot 2\cdot 3\cdot 4}\cdot a_0=\binom{\pi}{4}\cdot a_0
\end{array}$$

AWP: y(0) = 1 , also a_0 =1

$$y=1+\binom{\pi}{1}\cdot x+\binom{\pi}{2}\cdot x^2+\binom{\pi}{3}\cdot x^3+\binom{\pi}{4}\cdot x^4+\ldots=\sum_{k=0}^{\infty}\binom{\pi}{k}\cdot x^k.$$

A3:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\0&1&0\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}\qquad AWP:\begin{cases}x(0)=1\\y(0)=2\\z(0)=3\end{cases}$$

a) Charakteristische Gleichung:

$$P(\lambda)=\begin{Vmatrix}-\lambda&1&0\\0&-\lambda&1\\0&1&-\lambda\end{Vmatrix}=-\lambda\cdot\left(\lambda^2-1\right)=0\quad\Leftrightarrow\quad\lambda_1=0\,;\lambda_{2/3}=\pm1$$

$$\lambda_1=0:\quad\begin{pmatrix}0&1&0\\0&0&1\\0&1&0\end{pmatrix}\cdot\vec{a}=\vec{0}\quad\Rightarrow a_2=a_3=0;a_1\,\text{...beliebig}\quad\Rightarrow\vec{a}=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\overrightarrow{r_1(t)}=\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot e^0=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\lambda_2=1:\quad\begin{pmatrix}-1&1&0\\0&-1&1\\0&1&-1\end{pmatrix}\cdot\vec{a}=\vec{0}\quad\Rightarrow a_1=a_2\quad;a_3=a_2\quad\Rightarrow\quad\vec{a}=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\overrightarrow{r_2(t)}=\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot e^t$$

$$\overrightarrow{r_{allg}}(t) = C_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot e^t + C_3 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot e^{-t}$$

b) AWP:

$$\left.\begin{array}{l} x(t) = C_1 + C_2 \cdot e^t + C_3 \cdot e^{-t} \\ y(t) = C_2 \cdot e^t - C_3 \cdot e^{-t} \\ z(t) = C_2 \cdot e^t + C_3 \cdot e^{-t} \end{array}\right\} \Rightarrow \left.\begin{array}{rrrcl} C_1 & +C_2 & +C_3 & = & 1 \\ & C_2 & -C_3 & = & 2 \\ & C_2 & +C_3 & = & 3 \end{array}\right\} \Rightarrow$$

$$C_1 = -2; \quad C_2 = \frac{5}{2}; \quad C_3 = \frac{1}{2}$$

$$\left.\begin{array}{l} x_{spez}(t) = -2 + \frac{1}{2} \cdot \left(5e^t + e^{-t}\right) \\ y_{spez}(t) = \frac{1}{2} \cdot \left(5e^t - e^{-t}\right) \\ z_{spez}(t) = \frac{1}{2} \cdot \left(5e^t + e^{-t}\right) \end{array}\right\}$$

A4: $\vec{v} = \begin{pmatrix} x \cdot (x^2 - 3y^2) \\ y \cdot (y^2 - 3x^2) \end{pmatrix}$; $A(\frac{1}{2};2), B(1;1)$

a)

$div\, \vec{v} = 3x^2 - 3y^2 + 3y^2 - 3x^2 = 0 \quad \Leftrightarrow \vec{v}$ ist quellen- und senkenfrei

$$rot\, \vec{v} = \begin{Vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 - 3xy^2 & y^3 - 3x^2y & 0 \end{Vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6xy + 6xy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{v}$ ist wirbelfrei; $\vec{v}$ ist ein Gradientenfeld

b) Linienintegrale

1) $Weg\,(C_1)$: $y = \frac{1}{x} \Rightarrow \overrightarrow{r(t)} = \begin{pmatrix} t \\ \frac{1}{t} \end{pmatrix}$; $\overrightarrow{dr} = \begin{pmatrix} 1 \\ \frac{-1}{t^2} \end{pmatrix} dt$; $\frac{1}{2} \le t \le 1$

$$J_1 = \int_{\frac{1}{2}}^{1} \left(\left(t^3 - \frac{3t}{t^2} \right) + \left(-\frac{3t^2}{t} + \frac{1}{t^3} \right) \cdot \frac{-1}{t^2} \right) dt = \int_{\frac{1}{2}}^{1} \left(t^3 - \frac{1}{t^5} \right) dt = \left[\frac{t^4}{4} + \frac{1}{4t^4} \right]_{\frac{1}{2}}^{1} = -\frac{225}{64}$$

Nach der Punkt-Steigungsform einer Geraden gilt für den 2. Weg:

g: y - 1 = -2 (x - 1) , also y = - 2x + 3

$$2)\quad Weg\,(C_2):\quad \Rightarrow \overrightarrow{r(t)} = \begin{pmatrix} t \\ 3-2t \end{pmatrix} \quad ; \quad \overrightarrow{dr} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} dt \quad ; \quad \frac{1}{2} \le t \le 1$$

$$J_2 = \int_{\frac{1}{2}}^{1} \left(t \cdot \left(t^2 - 3 \cdot (3-2t)^2 \right) - 2 \cdot (3-2t) \cdot \left((3-2t)^2 - 3t^2 \right) \right) dt = \ldots =$$

$$\int_{\frac{1}{2}}^{1} \left(-7t^3 - 18t^2 + 81t - 54 \right) dt = \left[\frac{-7t^4}{4} - 6t^3 + \frac{81}{2} t^2 - 54t \right]_{\frac{1}{2}}^{1} = \ldots = -\frac{225}{64}$$

c) Skalarfeld u (x;y) als Gradientenfeld (siehe Teil a)):

$$\frac{\partial u}{\partial x} = x^3 - 3xy^2 \quad \Rightarrow \quad u(x;y) = \frac{x^4}{4} - \frac{3}{2} x^2 y^2 + \varphi(y)$$

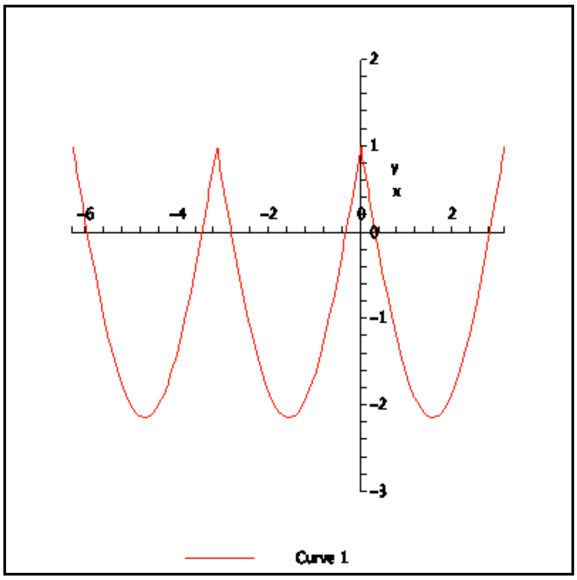
$$\Rightarrow \quad \frac{\partial u}{\partial y} = -3x^2 y + \varphi'(y) = -3x^2 y + y^3 \quad \Rightarrow \quad \varphi'(y) = y^3 \quad \Rightarrow \varphi(y) = \frac{y^4}{4} + C$$

$$also \quad u(x;y) = \frac{x^4}{4} - \frac{3}{2} x^2 y^2 + \frac{y^4}{4} + C = \frac{1}{4} \cdot \left(x^4 + y^4 \right) - \frac{3}{2} x^2 y^2 + C$$

$$u_{(1;1)} - u_{\left(\frac{1}{2};2\right)} = \frac{1}{2} - \frac{3}{2} - \frac{1}{4} \left(\frac{1}{16} + 16 \right) + \frac{3}{2} \cdot \frac{1}{4} \cdot 4 = -\frac{225}{64}.$$

A5: Fourier-Integral zu $f(x) = 1 - \pi \cdot |\sin x| = \begin{cases} 1 - \pi \cdot \sin x & 0 \le x \le \pi \\ 1 + \pi \cdot \sin x & -\pi \le x \le 0 \end{cases}$

a) Gerade Funktion, da f(- x) = f(x). Skizze:



$$a_0 = \frac{1}{\pi} \cdot \int_0^{\pi} (1 - \pi \cdot \sin x)\, dx = \frac{1}{\pi} \cdot \left[x + \pi \cdot \cos x \right]_0^{\pi} = -1$$

$$a_k = \frac{2}{\pi} \cdot \int_0^{\pi} (1 - \pi \cdot \sin x) \cdot \cos kx \, dx =$$

$$\frac{2}{\pi} \cdot \left\{ \underbrace{\left[\frac{\sin kx}{k} \right]_0^{\pi}}_{=0} + \pi \cdot \left[\frac{\cos(1+k)x}{2(1+k)} + \frac{\cos(1-k)x}{2(1-k)} \right]_0^{\pi} \right\}$$

$$= 2 \cdot \frac{1}{2} \cdot \left(\frac{\cos \overbrace{(1+k)\pi}^{\pi + k\pi}}{(1+k)} + \frac{\cos(1-k)\pi}{(1-k)} - \frac{1}{1+k} - \frac{1}{1-k} \right)$$

$$= \left(-\frac{(-1)^k}{1+k} - \frac{(-1)^k}{1+k} - \frac{1}{1+k} - \frac{1}{1-k} \right)$$

Voraussetzung: $k \ne 1$

$$a_1 = \frac{2}{\pi} \cdot \int_0^{\pi} (1 - \pi \cdot \sin x) \cdot \cos x \, dx = \left[\frac{1}{2} \cdot \cos 2x \right]_0^{\pi} = 0$$

$$a_k = \begin{cases} 0 & \text{für k ungerade} \\ -\frac{2}{1+k} - \frac{2}{1-k} = \frac{4}{k^2-1} & \text{für k gerade} \end{cases}$$

$$also: \quad f(x) = -1 + 4 \cdot \left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots \right)$$

c) x = 0 : f(0) = 1

$$1 = -1 + 4 \cdot \left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \frac{1}{63} + \ldots \right) \Leftrightarrow 2 = 4 \cdot \left(\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \ldots \right)$$

$$\left(\frac{2}{1 \cdot 3} + \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} + \frac{2}{7 \cdot 9} + \ldots \right) = 1$$

d) Parseval-Gleichung:

$$\underbrace{\frac{2}{\pi} \cdot \int_0^\pi f^2(x)\,dx}_{= \frac{2}{\pi} \cdot \int_0^\pi (1 - \pi \cdot \sin x)^2\,dx} = \underbrace{2a_0^2 + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2 \right)}_{= 2 + \sum_{k \text{ gerade}} a_k^2 = 2 + \sum_{k \text{ gerade}} \frac{16}{(k^2-1)^2}}$$

$$\frac{2}{\pi} \cdot \int_0^\pi \left(1 - 2\pi \cdot \sin x + \pi^2 \cdot \sin^2 x \right) dx = \frac{2}{\pi} \cdot \left[x + 2\pi \cdot \cos x + \frac{1}{2} \cdot \pi^2 \cdot \left(x - \sin x \cdot \cos x \right) \right]_0^\pi$$

$$= \frac{2}{\pi} \cdot \left(\pi - 2\pi + \frac{\pi^3}{2} - 2\pi \right) = \pi^2 - 6$$

$$\Rightarrow \frac{\pi^2 - 8}{16} = \frac{1}{3^2} + \frac{1}{15^2} + \frac{1}{35^2} + \frac{1}{63^2} + \ldots = \frac{1}{(1 \cdot 3)^2} + \frac{1}{(3 \cdot 5)^2} + \frac{1}{(5 \cdot 7)^2} + \ldots$$