A1) $y''' - 3y'' + 3y' - y = \{g(x)\} \quad mit \quad g(x) \in \{-6e^x, 8e^{2x}\cdot\cos x, x^2\cdot\sin x\}$

a) Homogene Lösung: $y = e^{\alpha x} \Rightarrow P(\alpha) = \alpha^3 - 3\alpha^2 + 3\alpha - 1 = (\alpha - 1)^3 = 0$

$\Rightarrow \alpha_{1/2/3} = 1 \quad \Rightarrow y_H = C_1 e^x + C_2 x\cdot e^x + C_3 x^2 \cdot e^x.$

Inhomogene Lösungen: $g_1(x) = -6e^x$; 3-fache Resonanz $\Rightarrow y_{P_1} = \dfrac{-6e^x\cdot x^3}{P'''(1)} =$

$= -x^3 \cdot e^x \quad \{P'(\alpha) = 3(\alpha-1)^2 \; ; \; P''(\alpha) = 6(\alpha - 1) \; ; P'''(\alpha) = 6.$

$g_2(x) = 8e^{2x}\cdot\cos x \quad ; \alpha = 2 + i \quad ;$ keine Resonanz $\Rightarrow y_{P_2} = \mathrm{Re}\left[\dfrac{8e^{(2+i)x}}{P(2+i)}\right] =$

$$= 8\cdot e^{2x}\cdot \mathrm{Re}\left[\frac{(\cos x + i\sin x)}{-2+2i}\cdot\frac{i+1}{i+1}\right] = -2e^{2x}\cdot(\cos x - \sin x)$$

$g_3(x) = x^2\cdot\sin x$ *keine* Resonanz: $\alpha = i \quad \Rightarrow$

$y_{p_3} = (Ax^2 + Bx + C)\cdot\cos x + (Dx^2 + Ex + F)\cdot\sin x$

allgemeine Lösung zu $g(x) = -6e^x + 8e^{2x}\cdot\cos x$:

$$y_{allg} = e^x\cdot(C_1 + C_2 x + C_3 x^2 - x^3) - 2e^{2x}\cdot(\cos x - \sin x)$$

b) Potenzreihenentwicklung:

$y(0) = a_0 = 2; \quad y'(0) = a_1 = 1; \quad y''(0) = a_2 = 0; \quad \Rightarrow y'''(0) = 0 - 3 + 2 - 6 + 8 = a_3 = 1$

$y^{(4)} - 3y''' + 3y'' - y' = -6e^x + 8e^{2x}\cdot(2\cos x - \sin x) \quad \Rightarrow y^{(4)}(0) = \ldots = a_4 = 14$

$y^{(5)} - 3y^{(4)} + 3y''' - y'' = -6e^x + 8e^{2x}\cdot(3\cos x - 4\sin x) \quad \Rightarrow y^{(5)}(0) = \ldots = a_5 = 57$

also: $y = 2 + x + \dfrac{x^3}{6} + \dfrac{7}{12}x^4 + \dfrac{19}{45}x^5 + \ldots$

Kontrolle für $x_0 = 1$: $y(1) \approx 2 + 1 + \dfrac{1}{6} + \dfrac{7}{12} + \dfrac{19}{45} \approx 4{,}225.$

A2) $y'' - \dfrac{3}{x}y' + \dfrac{4}{x^2}y = \dfrac{x-2}{x^4} \quad ; \quad x \geq 0.$

a) *Euler – Dgl*: $x^2\cdot y'' - 3x\cdot y' + 4y = \dfrac{x-2}{x^2} \quad \Rightarrow$ *Ansatz*: $y = x^\alpha$

$\Rightarrow P(\alpha) = \alpha^2 - \alpha - 3\alpha + 4 = (\alpha - 2)^2 = 0 \quad \Leftrightarrow \alpha_{1/2} = 2 \quad \Rightarrow y_H = C_1 x^2 + C_2 x^2\cdot\ln x$

b) *VdK*: $W = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & x\cdot(1 + 2\ln x) \end{vmatrix} = x^3 \neq 0.$

$$C_1'(x) = \frac{1}{x^3}\cdot\begin{vmatrix} 0 & x^2\ln x \\ \frac{x-2}{x^4} & x\cdot(1+2\ln x) \end{vmatrix} = -\frac{1}{x^4}\ln x + \frac{2}{x^5}\ln x$$

angegebene Formel für m=4 und m=5 ergibt: $C_1(x) = \dfrac{\ln x}{3x^3} + \dfrac{1}{9x^3} - \dfrac{\ln x}{2x^4} - \dfrac{1}{8x^4} + K_1$

$$C_2'(x)=\frac{1}{x^3}\cdot\begin{vmatrix} x^2 & 0 \\ 2x & \frac{x-2}{x^4}\end{vmatrix}=\frac{1}{x^4}-\frac{2}{x^5}\quad\Rightarrow\boxed{C_2(x)=\frac{-1}{3x^3}+\frac{1}{2x^4}+K_2}$$

$$\text{also:}\quad y_{\text{allg}}=\frac{\ln x}{3x}+\frac{1}{9x}-\frac{\ln x}{2x^2}-\frac{1}{8x^2}-\frac{x^2\ln x}{3x^3}+\frac{x^2\ln x}{2x^4}+K_1x^2+K_2x^2\ln x=$$

$$=\boxed{K_1x^2+K_2x^2\ln x+\underbrace{\frac{1}{9x}-\frac{1}{8x^2}}_{\text{hat die Form }\frac{A}{x}+\frac{B}{x^2}}}$$

A3) a) $\boxed{A=\begin{pmatrix}1&2&3\\2&-4&-2\\3&2&1\end{pmatrix}\quad;\quad\vec{y}'=A\cdot\vec{y}}$

$$\det(A-\lambda E)=\begin{vmatrix}1-\lambda&2&3\\2&-4-\lambda&-2\\3&2&1-\lambda\end{vmatrix}=\ldots=-(\lambda+4)\cdot\underbrace{\left(\left(\lambda^2-2\lambda-8\right)\right)}_{(\lambda+2)\cdot(\lambda-4)}=0$$

$$\Rightarrow\lambda_1=-4\quad;\quad\lambda_2=-2\quad;\quad\lambda_3=4\quad;$$

1) $\lambda_1=-4$: $\begin{pmatrix}5&2&3&0\\2&0&-2&0\\3&2&5&0\end{pmatrix}\sim\ldots\sim\begin{pmatrix}1&0&-1&0\\0&1&4&0\\0&0&0&0\end{pmatrix}\Rightarrow rg=2,\ n-rg=1$

also: $x_3=t\quad;\quad x_2=-4t\quad;\quad x_1=t\quad\Rightarrow\boxed{\vec{a}_1=\begin{pmatrix}1\\-4\\1\end{pmatrix}}$; 1.EV

2) $\lambda_2=-2$: $\begin{pmatrix}3&2&3&0\\2&-2&-2&0\\3&2&3&0\end{pmatrix}\sim\begin{pmatrix}1&4&5&0\\0&-10&-12&0\\0&0&0&0\end{pmatrix}\Rightarrow rg=2,\quad n-rg=1$

also: $x_3=10t\quad;\quad x_2=-12t\quad;\quad x_1=-2t\quad\Rightarrow\boxed{\vec{a}_2=\begin{pmatrix}-1\\-6\\5\end{pmatrix}}$; 2.EV

3) $\lambda_3=4$: $\begin{pmatrix}-3&2&3&0\\2&-8&-2&0\\3&2&-3&0\end{pmatrix}\sim\begin{pmatrix}1&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}\Rightarrow rg=2,\quad n-rg=1$

also: $x_3=t\quad;\quad x_2=0\quad;\quad x_1=t\quad\boxed{\vec{a}_3=\begin{pmatrix}1\\0\\1\end{pmatrix}}$; 3.EV

$$\Rightarrow\boxed{\vec{y}_{\text{allg}}=C_1\cdot\begin{pmatrix}1\\-4\\1\end{pmatrix}\cdot e^{-4x}+C_2\cdot\begin{pmatrix}-1\\-6\\5\end{pmatrix}\cdot e^{-2x}+C_3\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot e^{4x}}$$

$$\left.\begin{aligned}y_1&=C_1\cdot e^{-4x}-C_2\cdot e^{-2x}+C_3\cdot e^{4x}\\y_2&=-4C_1\cdot e^{-4x}-6C_2\cdot e^{-2x}\\y_3&=C_1\cdot e^{-4x}+5C_2\cdot e^{-2x}+C_3\cdot e^{4x}\end{aligned}\right\}\quad y_1'=y_1+2y_2+3y_3=\ldots=\boxed{-4C_1\cdot e^{-4x}+2C_2\cdot e^{-2x}+4C_3\cdot e^{4x}}$$

A4)

a) Vektorfeld: $\vec{v} = \begin{pmatrix} \frac{xy}{z} \\ \frac{x^2}{2z} + (y-z)^2 \\ -\frac{x^2 y}{2z^2} - (y-z)^2 \end{pmatrix}$; $z \neq 0$

$$rot\,\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{xy}{z} & \frac{x^2}{2z} + (y-z)^2 & -\frac{x^2 y}{2z^2} - (y-z)^2 \end{vmatrix} = \begin{pmatrix} -\frac{x^2}{2z^2} + \frac{x^2}{2z^2} \pm 2(y-z) \\ \frac{2x\,y}{2z^2} - \frac{2x\,y}{2z^2} \\ \frac{x}{z} - \frac{x}{z} \end{pmatrix} = \underline{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

also existiert Gradientenfeld Φ:

$$\frac{\partial \Phi}{\partial x} = \frac{xy}{z} \quad \Rightarrow \Phi(x,y,z) = \frac{x^2 y}{2z} + C(y,z)$$

$$\frac{\partial \Phi}{\partial y} = \frac{x^2}{2z} + \frac{\partial C}{\partial y} = \frac{x^2}{2z} + (y-z)^2 \quad \Rightarrow C(y,z) = \frac{1}{3}(y-z)^3 + C(z)$$

$$\frac{\partial \Phi}{\partial z} = -\frac{x^2 y}{2z^2} - (y-z)^2 + C'(z) = -\frac{x^2 y}{2z^2} - (y-z)^2 \quad \Rightarrow C(z) = C$$

$$\Rightarrow \boxed{\Phi(x,y,z) = \frac{x^2 y}{2z} + \frac{1}{3}(y-z)^3 + C}$$

b) Weg C parallel zu den Koordinatenachsen von $P_1(-2;0;3) \to P_2(0;3;3)$

$$C_{11}: \left.\begin{matrix} x = t-2 \\ y = 0 \\ z = 3 \end{matrix}\right\} \quad \vec{r_{11}}(t) = \begin{pmatrix} t-2 \\ 0 \\ 3 \end{pmatrix} \quad ; \quad d\vec{r_{11}}(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dt \quad ; \quad 0 \le t \le 2$$

$$J_{11} = \int_{t=0}^{2} 0\,dt = \underline{|0}$$

$$C_{12}: \left.\begin{matrix} x = 0 \\ y = t \\ z = 3 \end{matrix}\right\} \quad \vec{r_{12}}(t) = \begin{pmatrix} 0 \\ t \\ 3 \end{pmatrix} \quad ; \quad d\vec{r_{12}}(t) = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} dt \quad ; \quad 0 \le t \le 3$$

$$J_{12} = \int_{t=0}^{3} (t-3)^2\,dt = \left[\frac{(t-3)^3}{3}\right]_0^3 \underline{|9} \quad \Rightarrow J_{11} + J_{12} = \boxed{J = 9}$$

Kontrolle direkt mittels Φ: $\Phi(0;3,3) - \Phi(-2;0;3)) = C - \left(0 + \frac{1}{3}\cdot(-27) + C\right) = \boxed{9}$