A1)

a) $y''' - 4y' = 0 \quad \Rightarrow P(\alpha) = \alpha^3 - 4\alpha = 0 = \alpha \cdot (\alpha - 2) \cdot (\alpha + 2)$

$y_H = C_1 + C_2 \cdot e^{2x} + C_3 \cdot e^{-2x}.$

$b_1)$ $g_1(x) = 27x^2 \cdot e^x$; keine Resonanz, aber quadratischer Vorfaktor

Ansatz: $y_P = (Ax^2 + Bx + C) \cdot e^x \quad \Rightarrow \quad y_P' = (Ax^2 + (B+2A)x + (B+C)) \cdot e^x$

$y_P'' = (Ax^2 + (B+4A)x + (2A+2B+C)) \cdot e^x \quad y_P''' = (Ax^2 + (B+6A)x + (6A+3B+C)) \cdot e^x$

eingesetzt in Dgl:

$(Ax^2 + (B+6A)x + (6A+3B+C)) - 4(Ax^2 + (B+2A)x + (B+C)) = 27x^2$

$\Rightarrow A = -9; \quad B = 6; \quad C = -20. \quad \Rightarrow y_{spez} = C_1 + C_2 \cdot e^{2x} + C_3 \cdot e^{-2x} - e^x (9x^2 - 6x + 20)$

$b_2)$ AWP $y(0) = y'(0) = 2; \quad y''(0) = -2$

$\Rightarrow$ (1) $C_1 + C_2 + C_3 - 20 = 2$ (2) $2C_2 - 2C_3 - 14 = 2$ (3) $4C_2 + 4C_3 - 26 = -2$

$\Rightarrow C_1 = 16; \quad C_2 = 7; \quad C_3 = -1$; $y_{spez} = 16 - 7e^{2x} - e^{-2x} - e^x (9x^2 - 6x + 20)$

c) $g_2(x) = 27x^2 + r \cdot e^{rx}$

$g_{21}(x) = 27x^2 \quad \Rightarrow y_{P_1} = Ax^3 + Bx^2 + Cx$ wegen Resonanz zu α=0 !

in Dgl: $6A - 12Ax^2 - 8Bx - 4C = 27x^2 \quad \Rightarrow A = -\frac{9}{4}; \quad B = 0; \quad C = -\frac{27}{8}.$

$\Rightarrow \quad y_{P_1} = -\frac{9}{4}x^3 - \frac{27}{8}x = -\frac{9}{8}x \cdot (2x^2 + 3)$

$g_{22}(x) = r \cdot e^{rx}, \quad r > 0 \quad \Rightarrow$ einfache Resonanz zu $\alpha_2 = 2 \quad \Rightarrow y_{P_2} = \frac{2x \cdot e^{2x}}{P'(2)} = \frac{x \cdot e^{2x}}{4}$

$r \neq 2:$ keine Resonanz $\Rightarrow y_{P_2} = \frac{r \cdot e^{rx}}{P(r)} = \frac{e^{rx}}{r^2 - 4}.$

A2)

$x^2 \cdot y'' - 3x \cdot y' + 3y = x^3 - \frac{4}{x} \quad ; \quad x > 0$

$a_1)$ $x^2 \cdot y'' - 3x \cdot y' + 3y = 0$; *Ansatz* $y = x^\alpha \quad \Rightarrow P(\alpha) = \alpha(\alpha - 1) - 3\alpha + 3 =$

$\alpha^2 - 4\alpha + 3 = (\alpha - 1) \cdot (\alpha - 3) = 0 \quad \Rightarrow \quad y_H = C_1 \cdot x + C_2 \cdot x^3$

$a_2)$ $x^2 \cdot y'' - 3x \cdot y' + 4y = 0$; *Ansatz* $y = x^\alpha \quad \Rightarrow P(\alpha) = \alpha(\alpha - 1) - 3\alpha + 4 =$

$\alpha^2 - 4\alpha + 4 = (\alpha - 2)^2 = 0 \quad \Rightarrow \quad y_H = C_1 \cdot x^2 + C_2 \cdot x^2 \cdot \ln x$

b) VdK: $W(x) = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3 > 0$

Störfunktion: $g(x) = x - \frac{4}{x^3}$ wegen Normierung "1" vor y'' !!

$$C_1'(x)=\frac{1}{2x^3}\cdot\begin{vmatrix}0 & x^3\\ x-\frac{4}{x^3} & 3x^2\end{vmatrix}=-\frac{1}{2}x+\frac{2}{x^3}\quad\Rightarrow \boxed{C_1(x)=-\frac{1}{4}x^2-\frac{1}{x^2}+K_1}$$

$$C_2'(x)=\frac{1}{2x^3}\cdot\begin{vmatrix}x & 0\\ 1 & x-\frac{4}{x^3}\end{vmatrix}=\frac{1}{2x}-\frac{2}{x^5}\quad\Rightarrow \boxed{C_2(x)=\frac{1}{2}\ln x+\frac{1}{2x^4}+K_2}$$

also gilt: $y_{allg}=\left(-\frac{1}{4}x^2-\frac{1}{x^2}+K_1\right)\cdot x+\left(\frac{1}{2}\ln x+\frac{1}{2x^4}+K_2\right)\cdot x^3=$

$$=\boxed{\underbrace{K_1x+K_2x^3}_{y_H}\underbrace{-\frac{1}{2x}+\frac{x^3\ln x}{2}-\frac{x_3}{4}}_{y_P}}$$

A3)

a) $\boxed{\begin{array}{l}\dot{x}(t)=3x-2y+\sin t-9\cos t-9t+5\\ \dot{y}(t)=x+y-2\sin t+7\cos t-3t-1\end{array}}\quad\Rightarrow \ddot{y}=\dot{x}+\dot{y}-2\cos t-7\sin t-3=$

$=3x-2y+\sin t-9\cos t-9t+5+\dot{y}-2\cos t-7\sin t-3=$

$=3\dot{y}-3y+6\sin t-21\cos t+9t+3-2y+\sin t-9\cos t-9t+5+\dot{y}-2\cos t-7\sin t-3=$

$=4\dot{y}-5y-32\cos t+5\quad\Leftrightarrow\quad \boxed{\ddot{y}-4\dot{y}+5y=5-32\cos t}$

Homogene Lösung: $y=e^{\alpha t}\quad\Rightarrow P(\alpha)=\alpha^2-4\alpha+5=0\quad\Leftrightarrow(\alpha-2)^2=-1\quad\Leftrightarrow \boxed{\alpha_{1/2}=2\pm i}$

$\boxed{y_H(t)=e^{2t}\cdot(C_1\cos t+C_2\sin t)}$

b) Inhomogene Lösung: $g_1(t)=-32\cos t$; $\boxed{c=i,}$ also keine Resonanz!

$g_2(t)=5\quad\Rightarrow y_{P_2}=A\quad\Rightarrow 5A=5\quad\Leftrightarrow\quad\boxed{y_{P_2}=1}$

$$y_{P_1}=\mathrm{Re}\left[\frac{-32(\cos t+i\cdot\sin t)}{P(i)}\right]=\mathrm{Re}\left[\frac{-32(\cos t+i\cdot\sin t)\cdot(1+i)}{4(1-i)\cdot(1+i)}\right]=\boxed{-4(\cos t-\sin t)}$$

$\Rightarrow \boxed{y_{allg}(t)=e^{2t}\cdot(C_1\cos t+C_2\sin t)-4(\cos t-\sin t)+1}$

$x(t)=\dot{y}-y+2\sin t-7\cos t+3t+1=e^{2t}\cdot((2C_1+C_2)\cos t+(2C_2-C_1)\sin t)+4(\cos t+\sin t)$

$-e^{2t}\cdot(C_1\cos t+C_2\sin t)+4(\cos t-\sin t)-1+2\sin t-7\cos t+3t+1=$

$=\boxed{e^{2t}\cdot((C_1+C_2)\cos t+(C_2-C_1)\sin t)+2\sin t+\cos t+3t}$

In Vektor-Schreibweise:

$$\boxed{\overrightarrow{x(t)}=C_1e^{2t}\cdot\begin{pmatrix}\cos t-\sin t\\ \cos t\end{pmatrix}+C_2e^{2t}\cdot\begin{pmatrix}\cos t+\sin t\\ \sin t\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}\cdot\sin t+\begin{pmatrix}1\\-4\end{pmatrix}\cdot\cos t+\begin{pmatrix}3\\0\end{pmatrix}\cdot t+\begin{pmatrix}0\\1\end{pmatrix}.}$$