Lösung zu SS12:

A1) **a)** $y_1 = e^x \quad (\rightarrow \alpha_1 = 1)$, $\quad y_2 = e^x \cdot \cos x$ bzw. $y_3 = e^x \cdot \sin x \quad \left(\rightarrow \alpha_{2/3} = 1 \pm i\right)$

$$\Leftrightarrow P(\alpha) = (\alpha - 1) \cdot \underbrace{\{(\alpha - 1) + i\} \cdot \{(\alpha - 1) - i\}}_{=\alpha^2 - 2\alpha + 2} = \alpha^3 - 3\alpha^2 + 4\alpha - 2 = 0$$

$$\Leftrightarrow \boxed{y''' - 3y'' + 4y' - 2y = 0}$$

b1) $y''' - 3y'' + 4y' - 2y = e^x (3 - 4 \sin x)$

$g_{11}(x) = 3e^x \quad \rightarrow$ einfache Resonanz zu $\alpha_1 = 1 \qquad \Rightarrow \boxed{y_{P_1} = \frac{x \cdot 3e^x}{P'(1)} = \frac{x \cdot 3e^x}{1}}$

$g_{12}(x) = -4e^x \cdot \sin x \quad \rightarrow$ einfache Resonanz zu $\alpha_2 = 1 + i$

$$\Rightarrow \boxed{y_{P_2} = \mathrm{Im}\left[\frac{-4x \cdot e^x \cdot e^{ix}}{\underbrace{P'(1+i)}_{=-2}}\right]} = 2x \cdot e^x \cdot \sin x$$

also: $\quad y_{allg} = C_1 \cdot e^x + e^x \cdot (C_2 \cos x + C_3 \sin x) + 3x \cdot e^x + 2x \cdot e^x \sin x =$

$$= \boxed{e^x (C_1 + C_2 \cos x + C_3 \sin x + 3x + 2x \sin x)}$$

b2) $y''' - 3y'' + 4y' - 2y = \sqrt{e^{2x} \cdot 2x^2} = \sqrt{2} \cdot x \cdot e^x \quad \rightarrow$ Resonanz zu $\alpha_1 = 1$

$$\left.\begin{aligned}
&\Leftrightarrow y_{P_3} = x \cdot (Ax + B) \cdot e^x = (Ax^2 + Bx) \cdot e^x \\
&y'_{P_3} = (Ax^2 + x(B + 2A) + B) \cdot e^x \\
&y''_{P_3} = (Ax^2 + x(B + 4A) + 2(B + A)) \cdot e^x \\
&y'''_{P_3} = (Ax^2 + x(B + 6A) + 3(B + 2A)) \cdot e^x
\end{aligned}\right\} \text{ in Dgl eingesetzt:}$$

$$Ax^2 + x(B + 6A) + 3(B + 2A) - 3 \cdot (Ax^2 + x(B + 4A) + 2(B + A)) +$$
$$+4 \cdot (Ax^2 + x(B + 2A) + B) - 2 \cdot (Ax^2 + Bx) \doteq \sqrt{2} \cdot x$$

$$\Leftrightarrow \quad 2A \cdot x + B \doteq \sqrt{2} \cdot x \quad \Leftrightarrow \quad \boxed{A = \frac{\sqrt{2}}{2}; \quad B = 0} \quad \Leftrightarrow \boxed{y_{P_3} = \frac{\sqrt{2}}{2} x^2 e^x}$$

c1) AWP: $\; y(0) = 1 \; ; \; y'(0) = 2 \; ; \; y''(0) = 3$:

$(1) \quad 1 = C_1 + C_2$

$$y' = e^x \begin{pmatrix} C_1 + C_2 \cos x + C_3 \sin x + 3x + 2x \sin x + C_3 \cos x - C_2 \sin x + 3 \\ +2x \cos x + 2 \sin x \end{pmatrix}$$

$$(2)\quad 2=\underbrace{C_1+C_2}_{=1}+C_3+3 \quad\Rightarrow\quad \boxed{C_3=-2}$$

$$y''=e^x\begin{pmatrix} C_1+(C_2+C_3)\cos x+(C_3-C_2)\sin x+3x+2x\sin x+3+2x\cos x+2\sin x \\ -(C_2+C_3)\sin x+(C_3-C_2)\cos x+3+2x\cos x+2\sin x+4\cos x-2x\sin x \end{pmatrix}$$

$$(3)\quad 3=C_1+C_2+C_3+6+C_3-C_2+4=C_1+6 \quad\Leftrightarrow\quad \boxed{C_1=-3 \quad ; \quad C_2=4}$$

$$\textit{also:}\quad y_{spez}=-3e^x+4x\cdot e^x\cos x-2e^x\sin x+3xe^x+2x\cdot e^x\sin x=$$

$$=e^x\left(3x-3+4\cos x+2\sin x\left(x-1\right)\right)=\boxed{e^x\left\{(x-1)\cdot(3+2\sin x)+4\cos x\right\}}$$

c2) Potenzreihe über Dgl-Ableitung:

$$\rightarrow y'''(0)=3+9-8+2=\boxed{6}$$

$$y^{(4)}-3y'''+4y''-2y'=e^x\left(3-4\sin x-4\cos x\right)$$

$$\rightarrow y^{(4)}(0)=18-12+4-1=\boxed{9}$$

$$y^{(5)}-3y^{(4)}+4y'''-2y''=e^x\left(3-8\cos x\right)$$

$$\rightarrow y^{(5)}(0)=27-24+6-5=\boxed{4}$$

$$y^{(6)}-3y^{(5)}+4y^{(4)}-2y'''=e^x\left(3-8\cos x+8\sin x\right)$$

$$\rightarrow y^{(6)}(0)=12-36+12-5=\boxed{-17}$$

$$\textit{also:}\quad \boxed{\begin{aligned} y_{spez}&\approx 1+\frac{2}{1!}x^1+\frac{3}{2!}x^2+\frac{6}{3!}x^3+\frac{9}{4!}x^4+\frac{4}{5!}x^5-\frac{17}{6!}x^6+\ldots \\ &=1+2x+\frac{3}{2}x^2+x^3+\frac{3}{8}x^4+\frac{1}{30}x^5-\frac{17}{720}x^6+\ldots \end{aligned}}$$

$$y_{spez}(1)\approx 5{,}8847 \quad ; \quad y_{exakt}(1)=e\cdot\left(4\cos 1\right)=5{,}87478....$$

Relativer Fehler: $\dfrac{5{,}8847-5{,}8748}{5{,}8748}\cdot 100\% \approx \boxed{0{,}17\%}$

A2) a)

$$A=\begin{pmatrix}5&3&2\\-6&-4&-4\\1&1&2\end{pmatrix} \Rightarrow \det(A-\lambda E)=\begin{vmatrix}5-\lambda&3&2\\-6&-4-\lambda&-4\\1&1&2-\lambda\end{vmatrix}=-\boxed{\lambda(\lambda-1)(\lambda-2)=0}$$

3 reelle Eigenwerte mit Vielfachheit $e_i=1$: $\lambda_1=0;\quad \lambda_2=1;\quad \lambda_3=2;$

$$\text{ad}\quad \lambda_1=0:\quad \begin{pmatrix}5&3&2\\-6&-4&-4\\1&1&2\end{pmatrix}\ \Big|\cdot(-5)\Big|\cdot 6 \sim \begin{pmatrix}1&1&2\\0&-2&-8\\0&2&8\end{pmatrix}\sim\begin{pmatrix}1&0&-2\\0&1&4\\0&0&0\end{pmatrix}$$

$$\Rightarrow \vec{a}_1=\begin{pmatrix}2\\-4\\1\end{pmatrix}\quad;\quad rg=2\quad;\quad n-rg=1\quad(1\,EV)$$

$$\text{ad}\quad \lambda_2=1:\quad \begin{pmatrix}4&3&2\\-6&-5&-4\\1&1&1\end{pmatrix}\ \Big|\cdot(-4)\Big|\cdot 6 \sim \begin{pmatrix}1&1&1\\0&-1&-2\\0&1&2\end{pmatrix}\sim\begin{pmatrix}1&0&-1\\0&1&2\\0&0&0\end{pmatrix}$$

$$\Rightarrow \vec{a}_2=\begin{pmatrix}1\\-2\\1\end{pmatrix}\quad;\quad rg=2\quad;\quad n-rg=1\quad(1\,EV)$$

$$\text{ad}\quad \lambda_3=2:\quad \begin{pmatrix}3&3&2\\-6&-6&-4\\1&1&0\end{pmatrix}\ \Big|\cdot(-3)\Big|\cdot 6 \sim \begin{pmatrix}1&1&0\\0&0&2\\0&0&-4\end{pmatrix}\sim\begin{pmatrix}1&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$\Rightarrow \vec{a}_3=\begin{pmatrix}1\\-1\\0\end{pmatrix}\quad;\quad rg=2\quad;\quad n-rg=1\quad(1\,EV)$$

$$\textit{Lösung:}\quad \boxed{\vec{y}_H=C_1\cdot\begin{pmatrix}2\\-4\\1\end{pmatrix}+C_2\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot e^{x}+C_3\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot e^{2x}}$$

b) Inhomogenes System:

$$\vec{y}'=A\cdot\vec{y}+\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot e^{2x}\quad\Leftrightarrow\quad \text{Resonanz zu } \lambda_3=2\quad\Leftrightarrow \vec{y}_P=\underbrace{\left(\vec{a}_3\cdot x+\vec{b}\right)}_{\text{lineares Polynom}}\cdot e^{2x}$$

$$\vec{y}_P'=\underbrace{\left(\overrightarrow{2a_3}\cdot x+2\vec{b}+\vec{a}_3\right)}\cdot e^{2x}\quad \text{in Dgl eingesetzt:}$$

$$\underbrace{\left(\overrightarrow{2a_3}\cdot x+2\vec{b}+\vec{a}_3\right)}\cdot e^{2x}=A\cdot\underbrace{\left(\vec{a}_3\cdot x+\vec{b}\right)}\cdot e^{2x}+\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot e^{2x}$$

Koeffizientenvergleich: $(1)\quad A\cdot\vec{a_3}=2\vec{a_3} \quad\Leftrightarrow (A-2E)\cdot\vec{a_3} \;=0\ldots$ erfüllt

$$(2)\quad 2\vec{b}+\vec{a_3}=A\cdot\vec{b}+\begin{pmatrix}1\\-1\\3\end{pmatrix} \quad\Leftrightarrow (A-2E)\cdot\vec{b}=\vec{a_3}-\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\-3\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}3&3&2&0\\-6&-6&-4&0\\1&1&0&-3\end{array}\right) \begin{array}{l}\\ |\cdot(-3)|\cdot 6\end{array} \sim \left(\begin{array}{ccc|c}1&1&0&-3\\0&0&2&9\\0&0&-4&-18\end{array}\right)\sim\left(\begin{array}{ccc|c}1&1&0&-3\\0&0&2&9\\0&0&0&0\end{array}\right)$$

$$\Rightarrow \vec{b}=\begin{pmatrix}-3\\0\\4{,}5\end{pmatrix} \quad ; \quad \vec{y_P}=\left[\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot x+\begin{pmatrix}-3\\0\\4{,}5\end{pmatrix}\right]\cdot e^{2x}=\boxed{\begin{pmatrix}x-3\\-x\\4{,}5\end{pmatrix}\cdot e^{2x}}$$

also $$\boxed{\vec{y}_{\text{allg}}=C_1\cdot\begin{pmatrix}2\\-4\\1\end{pmatrix}+C_2\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot e^x+C_3\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot e^{2x}+\begin{pmatrix}x-3\\-x\\4{,}5\end{pmatrix}\cdot e^{2x}}$$

Kontrolle für $\vec{y'_1}=5y_1+3y_2+2y_3+1\cdot e^{2x}$

$$y'_{1_{\text{allg}}}=C_2\cdot e^x+2C_3\cdot e^{2x}+e^{2x}(1+2x-6)=\underline{C_2\cdot e^x+2C_3\cdot e^{2x}+e^{2x}(2x-5)}$$

$$\begin{aligned}5y_1+3y_2+2y_3+1\cdot e^{2x} &= 10C_1+5C_2\cdot e^x+5C_3\cdot e^{2x}+5(x-3)\cdot e^{2x}\\ &\quad -12C_1-6C_2\cdot e^x-3C_3\cdot e^{2x}-3x\cdot e^{2x}\\ &\quad +2C_1+2C_2\cdot e^x \qquad\qquad +9\cdot e^{2x}+e^{2x}\\ &= C_2\cdot e^x+2C_3\cdot e^{2x}+e^{2x}\underbrace{(5x-15-3x+9+1)}_{=2x-5} \quad\square\end{aligned}$$

A3) a)

$$\vec{v}=\begin{pmatrix}3x^2\cdot\ln(y+1)\\ \dfrac{x^3}{y+1}+z^2\\ 2yz+\sin z\end{pmatrix} \quad ; \quad \text{div }\vec{v}=6x\cdot\ln(y+1)-\frac{x^3}{(y+1)^2}+2y+\cos z \quad \ldots$$

es existieren Quellen und/oder Senken

$$\text{rot }\vec{v}=\begin{pmatrix}2z-2z\\0-0\\ \dfrac{3x^2}{y+1}-3x^2\cdot\dfrac{1}{y+1}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$ ... wirbelfrei. Gradientenfeld existiert!

grad $\vec{v}$ … nicht definiert

b) Weg C_1 :

$$\vec{r_1(t)}=\begin{pmatrix}t\\t\\1\end{pmatrix} \quad ; \quad \vec{dr_1(t)}=\begin{pmatrix}1\\1\\0\end{pmatrix}dt \quad ; \quad 0\le t\le 1$$

$$J_1 = \int_0^1 \left(3t^2 \cdot \ln(t+1) + \frac{t^3}{t+1} + 1 \right) dt = \left[t^3 \cdot \ln(t+1) + t \right]_0^1 = \underline{1 + \ln 2}$$

$$\text{Weg } C_2: \quad \overrightarrow{r_2(t)} = \begin{pmatrix} t \\ t^2 \\ 1 \end{pmatrix} \quad ; \quad \overrightarrow{dr_2(t)} = \begin{pmatrix} 1 \\ 2t \\ 0 \end{pmatrix} dt \quad ; \quad 0 \le t \le 1$$

$$J_2 = \int_0^1 \left(3t^2 \cdot \ln(t^2+1) + 2t \cdot \left(\frac{t^3}{t^2+1} + 1 \right) \right) dt = \left[t^3 \cdot \ln(t^2+1) + t^2 \right]_0^1 = \underline{1 + \ln 2}$$

c) Gradientenfeld u(x,y,z) :

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x} = 3x^2 \cdot \ln(y+1) \quad \Rightarrow u(x,y,z) = x^3 \cdot \ln(y+1) + C(y,z) \\
&\frac{\partial u}{\partial y} = \frac{x^3}{y+1} + \frac{\partial C}{\partial y} \doteq \frac{x^3}{y+1} + z^2 \quad \Rightarrow C(y,z) = y \cdot z^2 + C(z) \\
&\frac{\partial u}{\partial z} = 2yz + C'(z) \doteq 2yz + \sin z \quad \Rightarrow C(z) = -\cos z + K
\end{aligned}\right\}$$

$$\Leftrightarrow \boxed{u(x,y,z) = x^3 \cdot \ln(y+1) + y \cdot z^2 - \cos z + K}$$

$$u(1,1,1) - u(0,0,1) = 1 \cdot \ln 2 + 1 - \cos 1 + K - (0 + 0 - \cos 1 + K) = \underline{1 + \ln 2} \quad \square$$